ON THE THEORY OF PULSE DISCHARGE IN A LIQUID

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The time dependence of the electric power of underwater discharges is nearly linear during the first quarter-period. This paper presents the equations of energy, particle number, and channel expansion rate under this condition. It is shown that there exists a steady regime of channel expansion and shock propagation with constant characteristic properties, and the values of these properties are found.

1. The occurrence of a pulse discharge in a liquid is accompanied by the penetration of liquid particles into the arc channel. The channel constitutes a system with a variable number of particles. This is indicated by investigations of the underwater explosion of electrical wires [1], as well as by the fact that the pressure inside the expanding channel remains constant for a certain time, the temperature of the plasma changing insignificantly [2].

The penetration of the liquid particles into the channel is associated with the heating of the liquid at the periphery of the channel. This heating is mainly due to collisions between plasma and liquid particles; the contribution of radiation and three-particle recombinations cannot be significant. Due to the heating there appears between the plasma and the liquid a gas layer which loses particles to the channel, where these undergo further heating, dissociation, and partial ionization.

The rate at which particles penetrate into the channel is proportional to the rate of energy transfer by collisions at the periphery of the channel, and is inversely proportional to the energy of formation of the gas per particle. The rate of transfer of energy from the i-th component of the plasma is

$$\varepsilon_i' = \frac{N_i u_i \Delta\varepsilon_i}{2a}, \qquad \Delta\varepsilon_i = \frac{4mm_i kT}{(m+m_i)^2}. \qquad (1.1)$$

Here N_i is the number of particles of the i-th component, u_i is their mean thermal speed, m and m_i are the masses of a liquid molecule and a plasma particle, respectively, a is the channel radius, and $\Delta\varepsilon_i$ is the mean amount of energy transferred during one collision. From (1.1) and the gas-kinetic formula $z = {}^1/_4 uN/V$, which determines the number of collisions of the molecules with a unit area per unit time, we obtain the rate of penetration of the particles into the channel

$$N' = \frac{\varkappa\varepsilon'}{q} = 4\left(\frac{2}{\pi}\right)^{1/2} \frac{\varkappa m k^{3/2} N T^{3/2}}{qa} \sum_i \frac{\nu_i m_i^{1/2}}{(m+m_i)^2}, \qquad (1.2)$$

where q is the energy of formation of the gas per particle.

The theoretical calculation of the coefficient $\varkappa$ is unreliable, since it involves quite arbitrary assumptions. However, this coefficient can be found from any set of experimental data which can be used to draw a discharge power curve and to determine a characteristic of the channel. Using the experimental data of Skvortsov et al. [2], we obtain $\varkappa = 1/24$.

2. The energy supplied by the electrical circuit to the underwater spark channel goes towards the increase of the internal energy of the channel, the formation of the shock wave, and radiation; the radiation losses are minor. Analysis of current and voltage oscillograms of the discharge indicates that during the first quarter-period the time-dependence of the power dissipated is linear [3]:

$$w_e = \gamma t. \qquad (2.1)$$

At moderate degrees of ionization the mean energy of a plasma particle is

$$\varepsilon = {}^3/_2 kT + \varepsilon_d / \nu,$$

where ν is the number of atoms in a molecule of the liquid and ε_d is the dissociation energy per molecule. The change of the internal energy of the channel per unit time is then

$$w_i = (N\varepsilon)' = {}^3/_2 kN'T + {}^3/_2 kNT' + N'\varepsilon_d / \nu. \qquad (2.2)$$

The power transmitted to the shock wave is

$$w_g = \frac{NkTV_a'}{V_a} = \frac{2ka'NT}{a}, \qquad (2.3)$$

where a' is the speed of expansion of the channel. From shock theory it can be shown that one half of this power is expended in compressing the liquid and the other half is expended in setting it in motion. Equations (1.2), (2.1), (2.2), and (2.3) yield

$$N' = \frac{\nu\gamma}{\varepsilon_d} t - \frac{3k\nu}{2\varepsilon_d}(NT)' - \frac{2k\nu}{\varepsilon_d}\frac{a'}{a}(NT),$$

$$N'N^{1/2} = \frac{1}{6}\left(\frac{2}{\pi}\right)^{1/2} \frac{mk^{3/2}}{q} \sum_i \frac{\nu_i m_i^{1/2}}{(m+m_i)^2} \frac{(NT)^{3/2}}{a}. \qquad (2.4)$$

The system (2.4) contains three unknown functions (N, NT, a) and can be closed by the equations of hydrodynamics.

3. Due to their high nonlinearity, the equations of hydrodynamics cannot be used here in their general form, with boundary conditions at the channel boundary and at the shock front. One simplification is provided by the experimental fact that the speed of expansion of the channel is constant during the first quarter-period [2]. The effect of the expanding channel on the liquid is the same as that of an expanding cylindrical piston. Self-similar problems involving an expanding piston have been treated by

several authors [4,6]. In the self-similar problems the transformation to dimensionless variables transforms the equations of hydrodynamics into a system of ordinary differential equations. However, these equations cannot be integrated analytically even in the case of constant piston speed. Thus another simplification is required, for which we shall use the incompressibility of the liquid between the channel and the shock wave. In this region the liquid is compressed by the effect of the shock wave and subsequent density changes may be neglected [2].

Assuming incompressibility, the integration of the hydrodynamic equations yields the pressure field

$$p = p_a + \frac{\rho_0}{1 - a^2/R^2}\left[\frac{a'^2}{2}\left(1 - \frac{a^2}{r^2}\right) + (aa'' + a'^2)\ln\frac{a}{r}\right], \quad (3.1)$$

where p_a is the pressure in the channel, ρ_0 is the density of the undisturbed liquid, and R is the radial coordinate of the shock front.

The motion of the channel boundary is directly related to the propagation of the shock front. The impulse transmitted by the channel to the surrounding liquid per unit time is equal to the change of momentum of the liquid between the channel and the shock wave. The momentum integral necessarily converges, since it extends only up to the shock front. The equation for the rate of change of momentum is

$$\frac{d}{dt}\int_a^R \frac{2\pi\rho_0 u r dr}{1 - a^2/R^2} = 2\pi a p_a, \quad (3.2)$$

where u is the velocity of the fluid particles. From (3.2) we have

$$p_a = \frac{\rho_0}{1 - a^2/R^2}\left[(aa'' + a'^2)\frac{1 - a/R}{a/R} + a'^2\frac{1 - a'/D}{a'/D}\right]. \quad (3.3)$$

The solution to the self-similar problem of the motion of a fluid pushed by a uniformly moving piston shows that the shock wave also propagates at a constant velocity and that the pressure at the shock front is constant. Experimental data on shocks generated by underwater explosions also show that the speed of propagation of the shock front is constant when the channel expands at a constant speed [2]. Thus, in (3.1) and (3.3) we may take $a'' = 0$, $a/R = a'/D$, so that after expanding the logarithm we obtain

$$p_f = p_a + \frac{\rho_0 a'^2}{2} - \frac{2\rho_0 a'^2}{(1 + a'/D)^2}, \qquad p_a = \frac{2\rho_0 a' D}{1 + a'/D}. \quad (3.4)$$

The first equation in (3.4) shows that when the channel expands with a constant speed the pressure at the shock wave is lower than that in the channel. From the Rankine-Hugoniot relations and the equation of continuity of an incompressible fluid we obtain

$$uD = \frac{p_f}{\rho_0}, \quad u = \frac{aa'}{R} = \frac{a'^2}{D}, \quad \text{or} \quad \frac{p_f}{\rho_0} = a'^2 \quad (3.5)$$

(u is the particle velocity at the wavefront).

Substituting (3.5) into (3.4), we obtain two quadratic equations for the wavefront speed D. Equating the coefficients of these equations, we obtain the equation for the speed of expansions of the channel:

$$a'^4 + \frac{p_a}{2\rho_0}a'^2 - \frac{p_a^2}{\rho_0^2} = 0, \quad \text{or} \quad a' = 0.8\left(\frac{p_a}{\rho_0}\right)^{1/2}.$$

4. The pressure in the channel can be written in the form

$$p_a = \frac{kNT}{\pi a^2 l}.$$

Hence, taking account of (3.6) and of the fact that the expansion speed is constant, we obtain

$$NT = \frac{1.6\pi\rho_0 l a'^4 t^2}{k}. \quad (4.1)$$

Solving (2.4) for NT and equating the result with (4.1), we obtain an equation for a':

$$\gamma\left\{5k + 2\frac{\varepsilon_d}{\nu}\left[\frac{1}{12}\left(\frac{2}{\pi}\right)^{1/2}\frac{mk^{3/2}}{qa'}\sum_i\frac{\nu_i m_i^{1/2}}{(m + m_i)^2}\right]^{2/3}\right\}^{-1} = \frac{1.6\pi\rho_0 l a'^4}{k}. \quad (4.2)$$

In the denominator of the left-hand side of (4.2) we can neglect the second term. Then

$$a' = \left(\frac{1}{8\pi\rho_0}\gamma_1\right)^{1/4} \qquad \left(\gamma_1 = \frac{\gamma}{l}\right), \quad (4.3)$$

Table 1

γ, w/sec	l, cm	γ_1, w/(sec m)	a', m/sec	T, °K	N particles, t = 5 μ sec	j_N, sec^{-1} cm^{-2}	n_* cm^{-3}	p_a, kg/cm^2
10^{12}	3	$3.3\cdot10^{13}$	188	10 000	$2.2\cdot10^{19}$	$5\cdot10^{24}$	$2.9\cdot10^{20}$	400
$5\cdot10^{12}$	3	$1.7\cdot10^{14}$	280	13 000	$9.3\cdot10^{19}$	$1.4\cdot10^{25}$	$5.4\cdot10^{20}$	1000
10^{13}	3	$3.3\cdot10^{14}$	330	14 600	$1.7\cdot10^{20}$	$2.2\cdot10^{25}$	$7.0\cdot10^{20}$	1500
$5\cdot10^{13}$	3	$1.7\cdot10^{15}$	500	19 000	$7.2\cdot10^{20}$	$5.9\cdot10^{25}$	$1.3\cdot10^{21}$	3600
10^{14}	3	$3.3\cdot10^{15}$	600	21 200	$1.3\cdot10^{21}$	$9.0\cdot10^{25}$	$1.7\cdot10^{21}$	5000

which, substituted into the solution of (2.4), yields the temperature of the plasma in the channel

$$T = f^{8/3}(\gamma_1)^{1/4}, \qquad f = 6.5q\left(mk^{3/2}\rho_0^{1/4}\sum \frac{\nu_1 m_i^{1/2}}{(m+m_i)^2}\right)^{-1}. \quad (4.4)$$

The last equation shows that the temperature of the plasma remains constant when the power supplied by the circuit to the channel increases linearly. For discharges commonly used in practice, the ratio γ_1 varies in the range $3 \cdot 10^{13}$–$3 \cdot 10^{15}$ W/(sec • m), which corresponds to temperatures 10^4–$2 \cdot 10^{4}$° K. By reducing the inductance of the circuit to 0.26 μH, Martin [1] has obtained the value $\gamma_1 = 1.3 \cdot 10^{16}$ W/(sec • m), which corresponds to a calculated temperature $2.72 \cdot 10^{10}$ K. The channel temperature is a relatively weak function of γ_1. This explains the fact the braking of the discharge by means of an inductance does not significantly reduce the temperature.

Table 2

γ_1, w/(sec m)	p_{ϕ}, kg/cm^2	D, m/sec	η_g, %
$3.3 \cdot 10^{13}$	364	1600	24.2
$1.7 \cdot 10^{14}$	820	1690	26.8
$3.3 \cdot 10^{14}$	1150	1750	27.4
$1.7 \cdot 10^{15}$	2620	2000	30.2
$3.3 \cdot 10^{15}$	3640	2140	30.4

The solution of (2.4) yields the number of particles in the channel

$$N = \frac{(\gamma_1)\, lt^2}{2\varepsilon_d/\nu + 5kf^{2/3}(\gamma_1)^{1/4}}, \quad (4.5)$$

and the particle flux density

$$j_N = \left(\frac{8\rho_0}{\pi^3}\right)^{1/4} \frac{(\gamma_1)^{3/4}}{2\varepsilon_d/\nu + 5kf^{2/3}(\gamma_1)^{1/4}}. \quad (4.6)$$

The flux density of the particles entering the channel is quite high, of the order of 10^{24}–10^{26} sec^{-1} × × cm^{-2}. From (4.3) and (4.5) we obtain the particle density in the channel

$$n = \frac{1.8\rho_0^{1/2}(\gamma_1)^{1/2}}{2\varepsilon_d/\nu + 5kf^{2/3}(\gamma_1)^{1/4}}. \quad (4.7)$$

According to (4.7), the particle density in the plasma is constant when the electric power increases linearly. The decrease of density due to the expansion of the channel is compensated by the influx of particles through the channel boundary. The particle density is of the order of 10^{20}–10^{21} cm^{-3} and is more dependent on the value of γ_1 than the temperature.

The pressure of the plasma is given by the expression

$$p_a = \frac{1.8\rho_0^{1/2}kf^{2/3}(\gamma_1)^{2/3}}{2\varepsilon_d/\nu + 5kf^{2/3}(\gamma_1)^{1/4}}. \quad (4.8)$$

For γ_1 = const the pressure inside the channel remains constant during the expansion process, due to the fact that the temperature and the particle density remain constant. Under ordinary conditions the pressure is of the order of 10^2–10^3kg/cm^2, with higher values in the case of stronger variation of the power pulse. The pressure is a stronger function of γ_1 than the particle density, and much stronger than the temperature. Therefore in the case of discharges braked by inductance low pressures are obtained with relatively high temperatures.

From (4.3)–(4.8) it follows that for constant rate of increase of electric power there exists a steady regime of channel expansion, with constant values of temperature, particle density, plasma pressure, and speed of expansion. Under these conditions the shock front propagates at a constant speed with a constant pressure.

Such a regime (or a similar one) is established by underwater sparks from the instant of formation of the underwater channel to the instant at which maximum electric power is reached. The shock front and the region next to it are formed during this period. The trapezoidal form of the pressure in the shock wave is due to the steady expansion regime.

5. For pulse discharge in water, substitution of the appropriate numerical values in (4.3)–(4.8) yields the following design formulas:

$$a' = 7.9 \cdot 10^{-2}\gamma_1^{1/4}\ \text{m/sec}, \quad T = 56\,\gamma_1^{1/4}\,°\text{K}, \quad (5.1)$$

$$N = \frac{\gamma_1 lt^2}{4.3 \cdot 10^{-19} + 3.9 \cdot 10^{-21}\gamma_1^{1/4}}\ \text{particles}, \quad (5.2)$$

$$j_N = \frac{4\gamma_1^{3/4}}{4.3 \cdot 10^{-19} + 3.9 \cdot 10^{-21}\gamma_1^{1/4}}\ \text{sec}^{-1}\text{m}^{-2}, \quad (5.3)$$

$$n = \frac{57\,\gamma_1^{1/2}}{4.3 \cdot 10^{-19} + 3.9 \cdot 10^{-21}\gamma_1^{1/4}}\ \text{m}^{-3}, \quad (5.4)$$

$$p_a = \frac{4.5 \cdot 10^{-20}\gamma_1^{2/3}}{4.3 \cdot 10^{-19} + 3.9 \cdot 10^{-21}\gamma_1^{1/4}}\ \text{N/m}^2. \quad (5.5)$$

In these formulas $[\gamma_1]$ = W/sec • m). Table 1 shows the results of calculations for several values of γ_1, appropriate for pulse devices with l = 3 cm.

Table 3

Ref.	Circuit parameters				γ, w/sec	γ_1, w(sec m)	Measured quantity	Exptl. value	Theor. value	Calc. from eq.
	k v	μf	μH	cm						
[1]	25	5.8	0,26	1.5	$1.9 \cdot 10^{14}$	$1.3 \cdot 10^{16}$	T, °K	30.000	27.200	(5.2)
[2]	40	2.7	7	1.5	$1.3 \cdot 10^{12}$	$8.6 \cdot 10^{13}$	a', m/sec	200	240	(5.1)
[3]	40	2.7	7	1.5	$1.3 \cdot 10^{12}$	$8.6 \cdot 10^{13}$	D, m/sec	1600	1630	(6.6)
[4]	6	150	2	7	2.10^{12}	$2.8 \cdot 10^{13}$	a', m/sec	140	180	(5.1)

6. The constancy of the values of the speed and the pressure of the shock wave is due to the flux of energy from the channel to the shock front through the compressed liquid.

From (3.5) and (4.3) we obtain the pressure at the shock front

$$p_{\mathrm{f}} = {}^1/_5\, \rho_0^{1/2} \gamma_1^{1/2}. \tag{6.1}$$

Substituting (3.5) into Kirkwood and Bethe's [7] equation for shock waves in liquids

$$D = c_0 + {}^1/_4\,(n + 1)\, u$$

yields an equation for the shock speed D. The solution of this equation is

$$D = \frac{c_0}{2}\left\{1 + \left[1 + \frac{0.4\,(n+1)}{\rho_0^{1/2} c_0^2}\gamma_1^{1/2}\right]^{1/2}\right\}, \quad c_0 = \left(\frac{\partial p}{\partial \rho}\right)_s^{1/2}. \tag{6.2}$$

Here c_0 is the speed of sound in the undisturbed fluid n is an exponent in the equation of state of the fluid.

According to (6.2), the speed of the shock wave during the steady channel expansion regime usually lies in the range 1600–2000 m/sec, slowly increasing with increasing γ_1.

After the electric power has reached its maximum, the values of the characteristic properties (T, n, p_d) decrease, and the energy transmitted to the shock front decreases, resulting in a decrease of its speed and pressure. The subsequent motion of the front is governed mainly by the dissipation of the energy of the wave.

When the characteristics of the channel and the pressure at the shock front during the steady expansion regime depend only on γ_1, then the pressure of the wave at some distance away from the channel depends also on the duration of the period of power increase τ.

In many cases the variation of the circuit parameters v, L, C, l may lead to opposite changes of the values of γ_1 and τ. In such cases the pressure at the wave far away from the channel will not undergo significant changes, despite the fact that the characteristics of the channel may change quite considerably.

Substituting into (6.1) and (6.2) the numerical values for water, we obtain

$$p_{\mathrm{f}} = 6.4\,\gamma_1^{1/2}\ \ \mathrm{N/m^2} \tag{6.3}$$

$$D = 7.5\cdot 10^2\,[1 + (1 + 4.4\cdot 10^{-8}\gamma_1^{1/2})^{1/2}]\ \mathrm{m/sec}. \tag{6.4}$$

7. Regarded as a mechanism for transforming electrical energy into shock wave energy, underwater sparks are characterized by the electro-hydrodynamic efficiency, defined as the ratio of the energy of the shock wave to the electrical energy supplied to the channel by the circuit.

Under the steady channel expansion regime $\gamma_1 = \mathrm{const}$, equations (2.1), (2.3), (4.4), and (4.5) yield the electrohydrodynamic efficiency

$$\eta_g = \frac{2kf^{2/3}\gamma_1^{1/4}}{2e_d/\nu + 5kf^{2/3}\gamma_1^{1/4}}. \tag{7.1}$$

According to (7.1), under ordinary conditions η_g is in the range 25–30%. In 2 it was mentioned that one half of the energy supplied to the shock wave takes the form of compression energy, and the other half takes the form of kinetic energy.

For underwater discharges equation (7.1) takes the form

$$\eta_g = \frac{1.6\cdot 10^{-21}\gamma_1^{1/4}}{4.3\cdot 10^{-19} + 3.9\cdot 10^{-21}\gamma_1^{1/4}}. \tag{7.2}$$

Equations (7.1) and (7.2) determine η_g only for the increasing part of the pulse and cannot be used to calculate the overall efficiency. Electrical energy is transformed into hydrodynamic energy throughout the whole period during which energy is supplied to the circuit. But in addition to that hydrodynamic energy is also created from part of the energy of the gas layer in the course of the processes occuring after the pulse discharge.

Table 2 shows the results of calculations according to (6.3), (6.4), and (7.2).

A system of equations for the rate of expansion of the channel and for the pressure in it was obtained by a different method by Ioffe et al. [8].

8. The theoretical calcualtions are compared with the experimental data of several investigators in Table 3.

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